Circular No. 38 1952

STATE OF ILLINOIS

On The Determination of Transmissibility and Storage Coefficients from Pumping Test Data

Issued by

Department of Registration and Education

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ON THE DETERMINATION OF TRANSMISSIBILITY AND STORAGE COEFFICIENTS FROM PUMPING TEST DATA

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Abstract--This paper presents a graphical procedure for determining the formation constants of an artesian aquifer from pumping test data. The procedure is based on the principle that the coefficient of transmissibility is determined by the ratio of the drawdown to its rate of change with respect to the logarithm of the time since pumping started, or $s/(\delta s/\delta \log_{10} t)$, use being made of the non-equilibrium theory. The computation of formation constants may be performed in two ways, and their results should be checked against each other. A numerical example is given to illustrate the application of the procedure.

Introduction--In quantitative studies of ground-water hydrology for the development of water resources in an artesian aquifer it is essential to determine the field values of transmissibility and storage coefficients since these two so-called "formation constants" are generally considered as the physical indexes of the aquifer characteristics, and are used as the bases for the theoretica ??ediction on of the future yield of ground water in storage. This fact has been strongly recognized ??applying the science of ground-water hydrology to engineering problems, particularly in the engineering office where, in order to determine the field values of these constants, the engineers art seeking reliable and simple procedures that are adaptable to the solution of practical problems.

The coefficient of transmissibility indicates the rate at which an aquifer will transmit water under a unit hydraulic gradient. It is defined by THEIS [1938, p. 894] as "the number of gallons of water which will move in one day through a vertical strip of the aquifer one foot wide and having the height of the aquifer when the hydraulic gradient is unity."

The coefficient of storage characterizes the ability of the aquifer to release water from storage as the head declines. It is defined by JACOB [1940, p. 576]as "the volume of water of a certain density released from storage within the column of aquifer underlying a unit-surface area during a decline of head of unity."

The field values of these two coefficients can be determined by the method of non-equilibrium theory developed by THEIS [1935], in which theory the drawdown produced by a well pumping at a constant rate from an areally extensive artesian aquifer of uniform thickness and physical properties is given by

$$s = (114.6\, Q/T)\, W(u) \quad (1)$$

where s = drawdown in feet, Q = constant discharge of the pumped well in gallons per minute, T = coefficient of transmissibility in gallons per day per foot, and W (u) is a so-called well function. The well function is an exponential integral and can be expanded in the form of an infinite series as

$$W(u) = \int_u^{\infty} (e^{-u}/u)\, du = -0.577216 - \log_e u + u - u^2/2\cdot 2! + u^3/3\cdot 3! - \ldots \quad (2)$$

in which e = base of natural logarithm and u is an argument expressed as

$$u = 1.87\, r^2\, S/Tt \quad (3)$$

where r = distance of the observation well from the pumped well in feet, S = coefficient of storage a ratio or fraction, and t = time since pumping began in days.

When the values of formation constants are given, the computations of drawdown may be made by (1), (2), and (3), and are even simpler using a nomograph [CHOW, 1951], or a special chart or slide rule. However, the reverse procedure would present real difficulties. Theoretically speaking, if the drawdowns at two or more observation points at given times, or the rate of change of drawdown at a given observation point, as available from the pumping test data, are known, then the values of formation constants, S and T, are determined by (1), (2), and (3). Yet, owing to the fact that one of the two unknowns, T, occurs twice in the equation, once as the divider of the exponential integral and once in the denominator of the argument, a direct mathematical solution is extremely difficult. A cut-and-try solution is theoretically possible, but due to the wide and uncertain range of estimation and the laboriousness of computation such solution is by no means practical. Within the author's knowledge, two graphical methods for the solution of (1), (2), and (3) for T and S have been proposed respectively by Theis and Jacob.

Theis has suggested to WENZEL [1942, p. 88] in 1937 and to JACOB [1940, p. 582] in 1938 a graphical method of superposition for the solution of S and T from (1), (2), and (3). By this method, values of drawdown, s, are plotted against values of r^2/t, or of $1/t$ if there is only one observation well involved, on logarithmic paper to the same scale as the "type curve," resulting in a curve of the observed data. The "type curve" is a curve showing the relation between the argument u and the function W (u). It can be proved that the curve of the observed data should be a certain portion of the type curve, bearing the same slope or the same rate of change of curvature. The curve of the observed data is then superimposed on the type curve in such a way that the coordinate axes of the two curves are kept in parallel. A position is found by trial whereby most of the plotted points of the observed data fall on a segment of the type curve. With the curves in the superimposed position, an arbitrary point is chosen on the segment of fitness. The corresponding coordinates of the match point on the two curves, W (u) and u on the type curve and s and r^2/t, or $1/t$, on the observed curve, are determined. Substituting these coordinate values in (1) and (3), the values of T and S can be computed.

JACOB [1946] has found that when the argument u in (1) is small, that is, r is small and t is large as shown by (3), a plot of the drawdown s against the logarithm of the elapsed time t since pumping began should approximate a straight line. From the slope of this straight line, or the drawdown difference per log cycle of time, As, and the time intercept on zero-drawdown axis, to, the formation constants, T and S, may be easily computed by the following two simple formulas

$$T = 264\, Q/\Delta s \qquad (4)$$

$$S = 0.3\, T\, t_0/r^2 \qquad (5)$$

Theis' method of superposition offers a very ingenious solution to the mathematical difficulties described previously. On the other hand, at the risky expense of accuracy, Jacob's method of approximation possesses a great advantage of simplicity. However, each method has its drawback when used alone. For the method of superposition, as the curvature of the type curve is very flat in the portion which is often most used in practical applications, the fitting of curves by trial is far from exact, and the resulting values obtained by this method would involve errors of personal judgment to a great extent. For the method of approximation, an erroneous answer is often reached when the plotted curve appears to be a straight line to the eye, while, in reality, it is far from the straight-line approximation. This latter case is commonly encountered when the time of pumping test is insufficient and consequently the test fails to produce data that could cover the region of approximation.

As a result of the above-mentioned drawbacks, the present practice is to use the method of approximation when the error involved in the straight-line approximation is tolerable, or to use the method of superposition when the error is significant. To follow this practice, the engineer must use his own judgment. Otherwise, he should be given a knowledge of the error that would be introduced by the application of the method of approximation to any given problem. This knowledge is to be presented in this paper.

Mathematically it can be shown that a semi-logarithmic plot of the relation of W (u) to u will display the same curvature of a semi-log plot of s versus t. The former plot will reveal that for values of u less than about 0.01 the plot practically approximates a straight line. Hence, this property is also true for the latter plot which is required by the method of approximation. By (24) and (25), to be derived later, for u less than 0.01 the method of approximation would introduce er-*TOTS* of less than one per cent in T and errors of less than four per cent in S. For the procedure described in this paper, a tolerable error of five per cent is assumed, and the corresponding value of u should be less than 0.05.

Moreover, experience has shown that the formation constants are gradually changing during the time of pumping. This phenomenon becomes evident particularly in the case of severe pumping or at the early stage of the pumping test, under which conditions the sudden release in pore water pressure would cause compression in formation that affects the realizable value of constants.

When this happens, it is difficult to fit properly the observed curve to any portion of a type curve. The procedure described in this paper, however, will help the computer to find the apparent value of formation constants at any instant as the pumping goes on.

Theory--From Theis' non-equilibrium theory (1), (2), and (3) are established. From Eq. (3)

$$\log_e u = \log_e (1.87\, r^2\, S/T) - \log_e t \quad \ldots\ldots (6)$$

Differentiating u, $\log_e$ u, W (u), and s of (3), (6), (2), and (1) with respect to $\log_e$ t, the resulting expressions are

$$\delta u/\delta \log_e t = -u \quad \ldots\ldots (7)$$

$$\delta \log_e u/\delta \log_e t = -1 \quad \ldots\ldots (8)$$

$$\delta W(u)/\delta \log_e t = -\delta \log_e u/\delta \log_e t + (\delta u/\delta \log_e t)(1 - u/2! + u^2/3! - \ldots)$$

or, from (7) and (8),

$$\left.\begin{aligned} \delta W(u)/\delta \log_e t &= 1 - u\,(1 - u/2! + u^2/3! - \ldots) \\ &= 1 - u + u^2/2! - u^3/3! + \ldots \\ &= e^{-u} \end{aligned}\right\} \quad \ldots\ldots (9)$$

$$\delta s/\delta \log_e t = (114.6\, Q/T)\; \delta W(u)/\delta \log_e t$$

or, from (9),

$$\delta s/\delta \log_e t = (114.6\, Q/T)\, e^{-u} \quad \ldots\ldots (10)$$

Converting the natural logarithm to the common logarithm, (10) becomes

$$\delta s/\delta \log_{10} t = (264\, Q/T)\, e^{-u} \quad \ldots\ldots (11)$$

Dividing (1) by (11),

$$s/(\delta s/\delta \log_{10} t) = W(u)\, e^{u}/2.3 \quad \ldots\ldots (12)$$

Let

$$F(u) = s/(\delta s/\delta \log_{10} t) \quad \ldots\ldots (13)$$

Then

$$F(u) = W(u)\, e^{u}/2.3 \quad \ldots\ldots (14)$$

Therefore, for a given value of u, the values of W (u) and F (u) can be computed by (2) and (14) respectively. Consequently, a relation between F (u), W (u), and u can be plotted as shown by Figures 1 and 2. Conversely, when F (u) is known, the values of W (u) and u can be found from these figures. Note on Figure i that W (u) is practically equal to 2.3 F (u) when W (u) is greater than 4.0, or when F (u) is greater than 1.74. From Figure 2, this corresponds to the condition that u is less than 0.01.

From (1) and (3), with values of W (u) and F (u) being given, the coefficients of transmissibility and storage can be computed by

$$T = 114.6\, Q\, W(u)/s \quad \ldots\ldots (15)$$

$$S = Ttu/1.87\, r^2 \quad \ldots\ldots (16)$$

When u is very small, that is, less than say 0.01, e^u approaches unity, and accordingly (14) becomes

$$F(u) = W(u)/2.3 \quad \ldots\ldots (17)$$

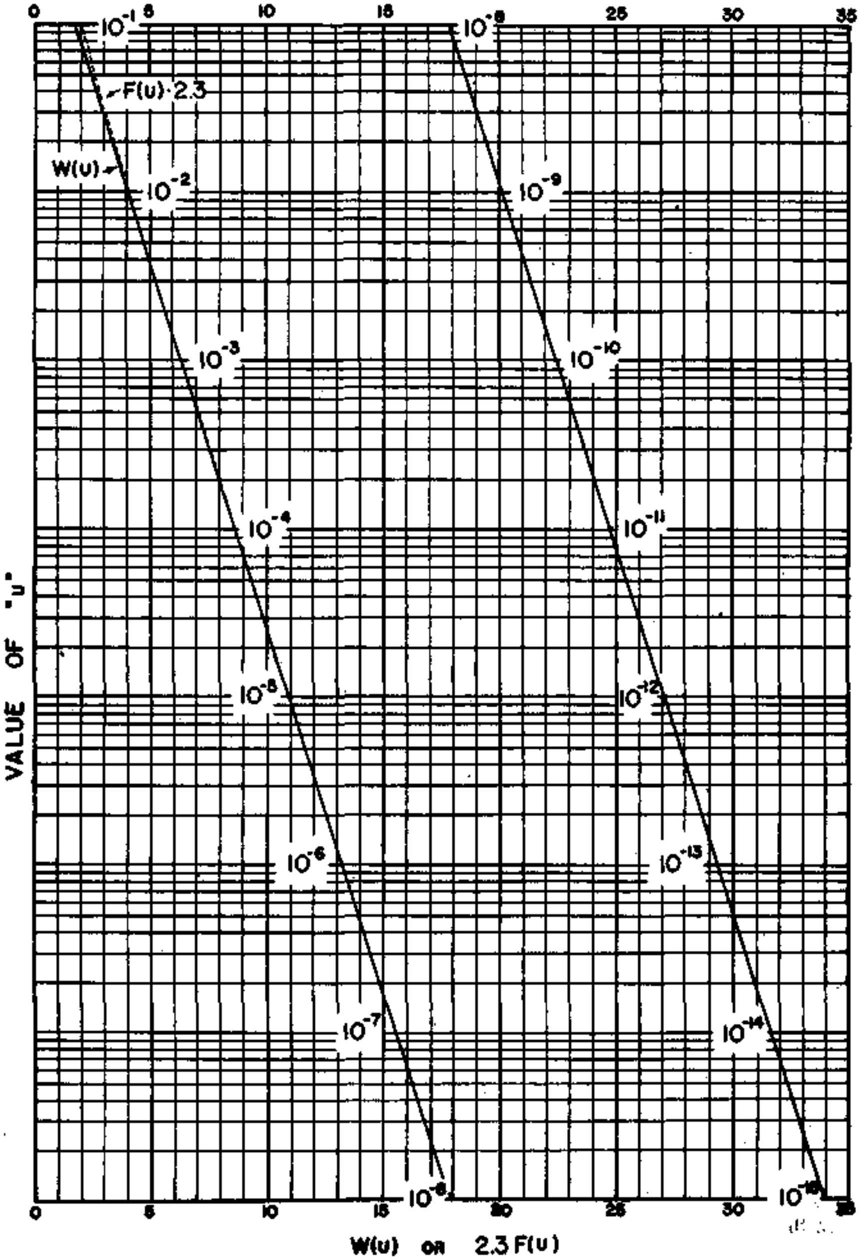

Fig. 1--Relation between W(u) or F(u) and u for small values of u

and (15) becomes

$$T = 264\ Q\ F(u)/s$$

or, from (13)

$$T = 264\ Q/(\partial s/\partial \log_{10} t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

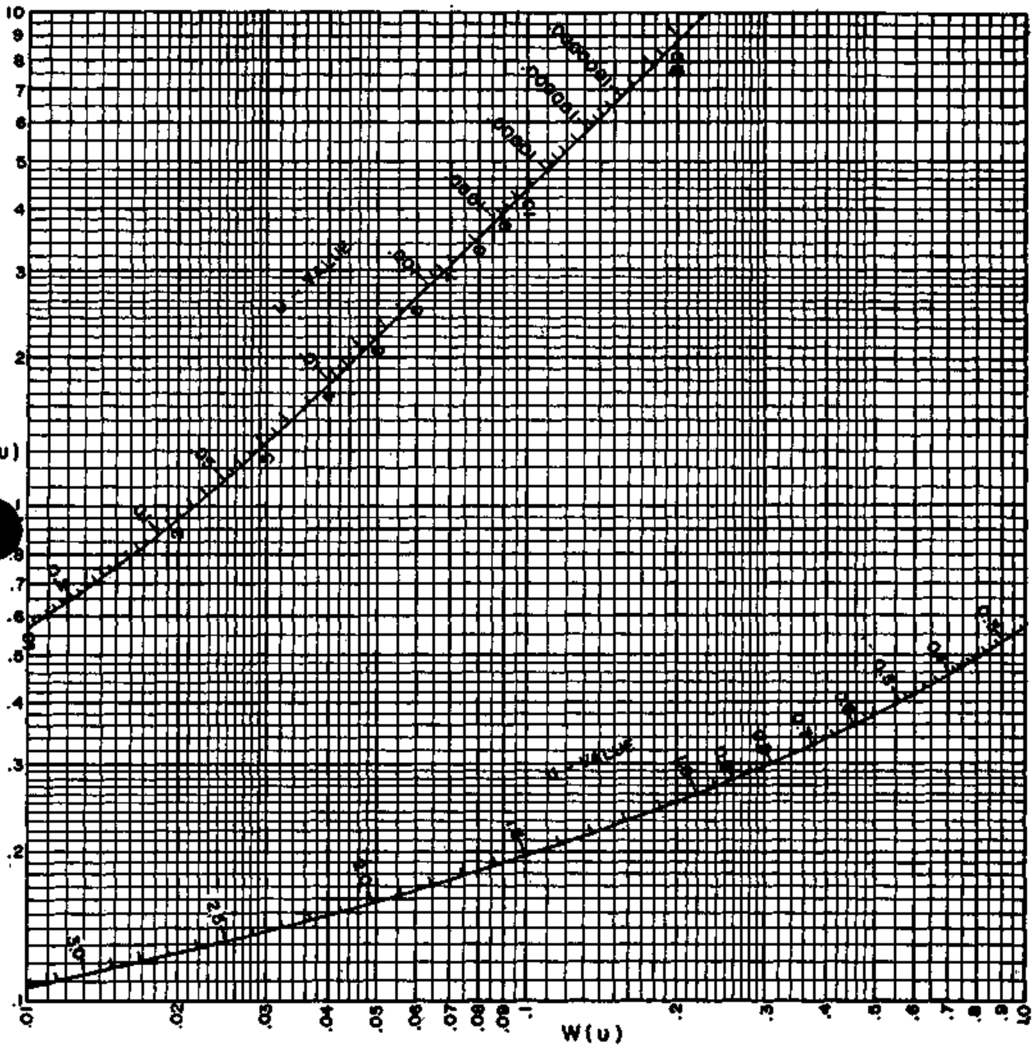

Fig. 2—Relation between F(u) and W(u) for large u-values

As mentioned previously, when u is very small, a plot of s against $\log_{10}$ t would approximate a straight line. The slope of this straight line is $\delta s/\delta \log_{10} t = \Delta s$. the drawdown difference per log cycle of time. Thus, (18) becomes (4).

Furthermore, when u is very small (less than 0.01), (2) approaches

$$W(u) = -0.577216 - \log_e u \quad (19)$$

Substituting (17) in (19) and transposing

$$u = e^{-0.577216-2.3F(u)} \quad (20)$$

Substituting (20) in (16) and transposing,

$$S = Tte^{-0.577216-2.3F(u)}/1.87\, r^2 \quad (21)$$

in which T is computed by (18) or (4).

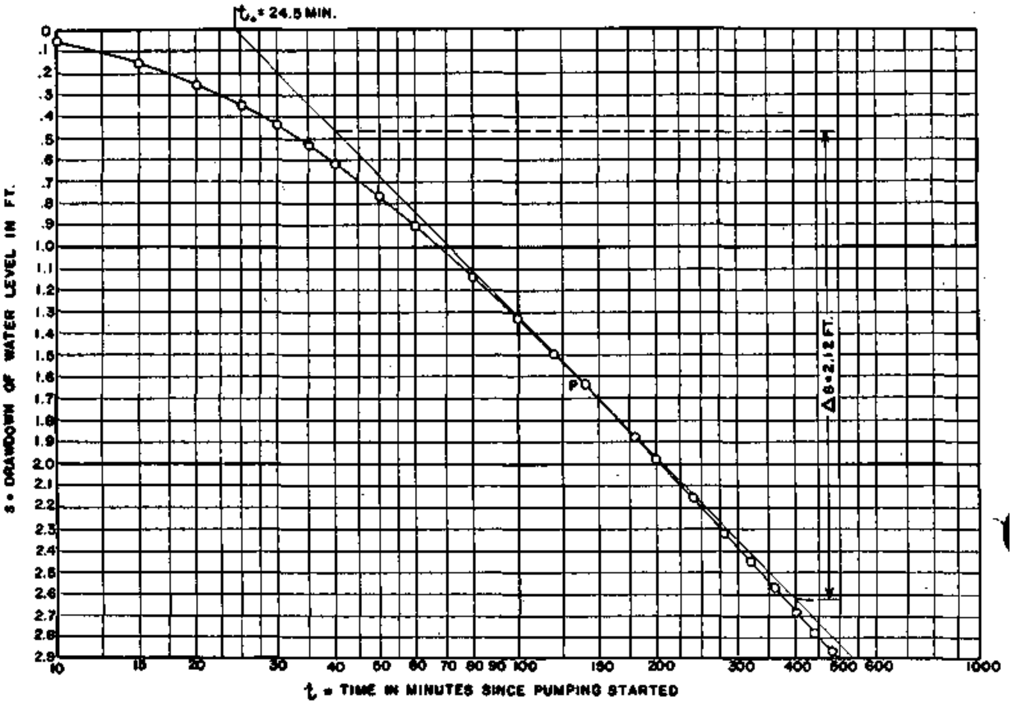

Fig. 3—Relation between drawdown and duration of pumping

From the straight-line assumption, or u being very small, when s = 0, t = t_0, which is the time intercept on zero-drawdown axis as shown by Figure 3, and from (18), F (u) = 0. Substituting these in (21) and simplifying, (21) becomes (5).

Equations (4) and (5), hence (18) and (21), are employed in Jacob's method of approximation. It can be proved that the relation between t_0 and t is

$$t_0/t = e^{-2.3F(u)} \quad \text{(22)}$$

and the relation between the approximate T computed by (4) and the T value computed by (15) is

$$T_{approx}/T = e^{u} \quad \text{(23)}$$

Consequently, the errors involved in the approximation for computing the values of T and S by (4) and (5) can be evaluated by means of (15), (16), (18), (21), (22), and (23) as follows

$$\text{The overestimated error in } T = e^{u} - 1 \quad \text{(24)}$$

$$\text{The underestimated error in } S = 1 - 0.561\, e^{u-2.3F(u)}/u \quad \text{(25)}$$

For given values of F (u) or u, the percentages of error involved in the computed T and S can be determined by (24) and (25) respectively. They are plotted against F (u) as shown by Figure 4. It is interesting to note that for an error of less than five per cent the values of F (u) should be restricted as follows: For an error of less than five per cent in T; F (u)>1.13. For an error of less than five per cent in S; F (u) > 1.62 and 0.23 > F (u) > 0.21. In other words, if an error of five per cent in transmissibility is tolerable, as generally accepted in such field methods, the method of approximation should be applied only when F (u) > 1.13.

Through the study and analysis of numerous pumping test data, the author has found that in most cases long-term test data are not available because the collection of such data is either uneconomical or beyond the interest of the well driller or the well owner. As a result, the value of F (u) derived from the available data is usually less than 1.13. Long-term test data could probably be secured only if it were the objective of a comprehensive investigation.

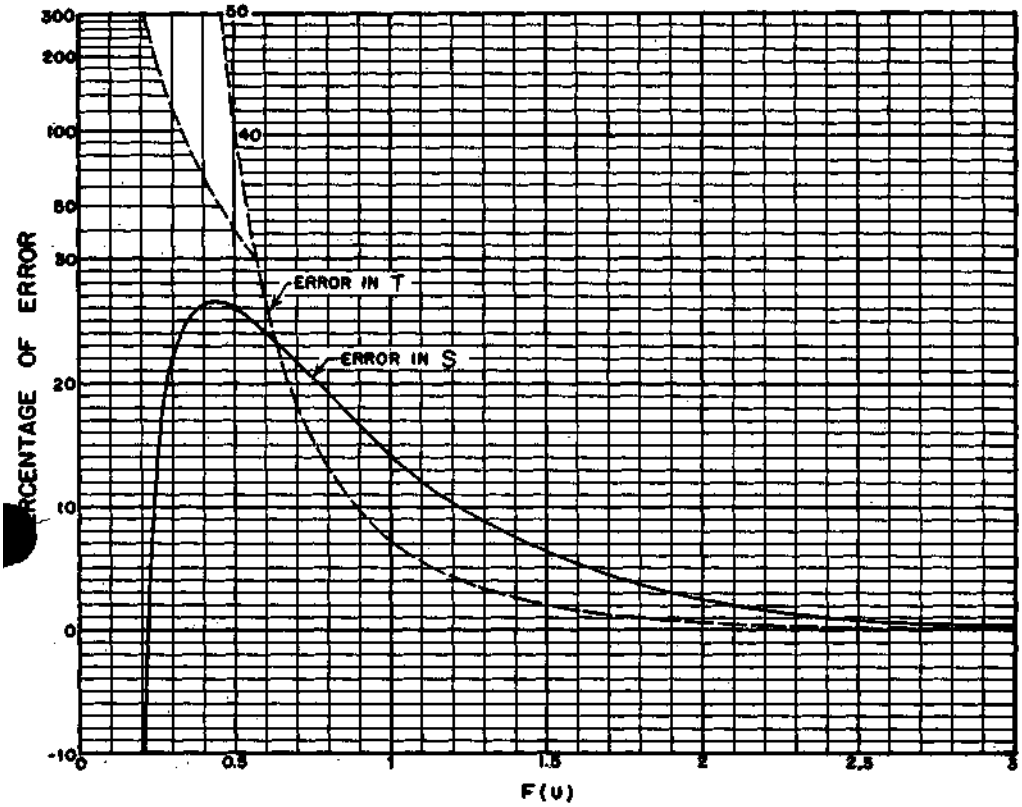

Fig. 4—Variation of per cent of error with F(u)

Procedure—The procedure for the application of the above theory to the determination of ormation constants from pumping test data may be outlined as follows:

(1) Plot the observational data for a well on semi-logarithmic coordinate paper in which the drawdown, s, in feet is in linear scale and the time, t, in minutes since pumping started, is in logarithmic scale.

(2) On the plotted curve, choose an arbitrary point and note the coordinate values, t and s.

(3) Draw a tangent to the curve at the chosen point and measure the slope of the tangent which is equal to $\delta s/\delta \log_{10} t$. For convenience, this slope may be taken as the drawdown difference per log cycle of time, or As, in feet. To construct a proper tangent may seem to be not easy for a beginner. However, with a little practice, the result should be a perfect job.

(4) The Computation of T and S can be performed in either of the following two methods:

(A) Compute F(u) by (13) and find the corresponding values of W(u) and u from Figure 1 or 2. Compute T and S by (15) and (16) respectively.

(B) Note the time intercept, to, on zero-drawdown axis, obtained by extending the tangent. Compute F(u) by (13) and find the percentages of error in T and S from Figure 4. Compute the approximate values of T and S by (4) and (5) respectively and correct the values for errors.

Example--A numerical example is given to illustrate the method of computation. The rate of pumping is 750 gpm. The distance of the observation well from the pumped well is 804 ft. As ??own in Figure 3, the drawdown, s, is plotted against the logarithm of time, $\log_{10} t$. A point P is ??sen on the curve. The coordinates of this point are found to be: t = 140 min = 0.0972 day; ?? = 1.64 ft. A tangent to. the curve at point P is constructed. The slope of this tangent expressed in drawdown difference per log cycle of time is As = 2.12 ft. The intercept of the tangent at zero-drawdown axis is t_0 = 24.5 min = 0.0170 day.

Method A—By (13), F (u) = 1.64/2.12 = 0.775. From Figure 2, W (u) = 1.55 and u = 0.135. Hence, by (15), T = 114.6 × 750 × 1.55/1.64 = 81,300 gpd/ft. By (16), S = 81,300 × 0.0972 × 0.135/1.87×804 × 804 = 8.83×10^{-4}.

Method B—By (13), F (u) = 0.775. From Figure 4, the overestimated error in T = 14.4 per cent and the underestimated error in S = 20.0 pet. The approximate T = 264 × 750/2.12 = 93,400 gpd/ft. The corrected T = 93.400/1.144 = 81,600 gpd/ft. The approximate S = 0.3 x 93,400 x 0.0170/804 × 804 = 7.38 × 10^{-4}. The corrected S = 7.38 × 10^{-4} × 1.2 = 8.86 × 10^{-4}.

Summary—The method for determining the formation constants as described in this paper has several significant features.

The computation of the formation constants can be performed in either of the two ways, Method A and Method B. The results of one method may be easily checked by the other.

The same s - $\log_{10}$ t plot used in the present procedure is readily applicable to the study of hydrologic boundaries of the ground-water reservoir [CHOW, BUSHMAN, and HUDSON, 1950], so that a more consistent and systematic procedure for the analysis of ground-water problems could be achieved.

A number of arbitrary points on the s - $\log_{10}$ t curve can be taken at various t. The values of T and S thus computed may be plotted against $\log_{10}$ t on the same coordinate paper resulting in T - $\log_{10}$ t and S - $\log_{10}$ t curves. As the non-equilibrium theory is based on the assumption of constant T and S, the theoretical T - $\log_{10}$ t and S - $\log_{10}$ t curves should appear to be horizontal straight lines. However, as often happens in practical problems, the values of T and S may be gradually increasing or decreasing, depending on the nature of the aquifer, during the pumping test. In such cases, the computed values are referred to as apparent values, the plot of which values against $\log_{10}$ t would give a clear picture showing the variation of aquifer characteristics with respect to the time of pumping.

For F (u) greater than 1.74, or u less than 0.01, the method of approximation gives practical the exact theoretical values of T and S. For F (u) less than 1.13, or u greater than 0.05, the meth of approximation should not be employed without tolerating an error of greater than five per cent.

The method of superposition involves the personal error from making trial solutions. Not uncommonly, a satisfactory fit between the observed curve and the type curve cannot be easily obtained, particularly when the apparent values of T and S are varying with the time of pumping at a remarkable degree.

Acknowledgments—The author is greatly indebted to J. J. Doland for his encouragement in pursuing this study, and to Engineering Subdivision, Illinois State Water Survey, for affording opportunities to investigate practical ground-water problems. The head of the Engineering Subdivision is H. E. Hudson, Jr.

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(Communicated manuscript received June 14, 1951; open for formal discussion until November 1, 1952.)